Year 12 Mathematics Specialist 2018

Test Number 1: Complex Numbers

Resource Free

Name: SOLUTIONS **Teacher: DDA**

Marks: **45**

Time Allowed: **45 minutes**

Instructions: You **ARE NOT** permitted any notes or calculator. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 1 **[4 marks]**

Solve the complex equation $z^4 = -16$.

$$z^4 = 16cis(\pi + 2k\pi)$$ ✓

$$\therefore z = (16cis(\pi + 2k\pi))^{\frac{1}{4}}$$

$$z = 2cis\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)$$ ✓

$k = 0 \quad \Rightarrow \quad z_1 = 2cis\frac{\pi}{4}$

$k = 1 \quad \Rightarrow \quad z_1 = 2cis\frac{3\pi}{4}$

$k = 2 \quad \Rightarrow \quad z_1 = 2cis\frac{5\pi}{4} = 2cis\left(-\frac{3\pi}{4}\right)$

$k = 3 \quad \Rightarrow \quad z_1 = 2cis\frac{7\pi}{4} = 2cis\left(-\frac{\pi}{4}\right)$ ✓ ✓

Question 2 **[3 marks]**

Given $P(x)=x^3+x^2+x-3$ find x such that $P(x)=0$ and hence solve the equation $x^3+x^2+x-3=0$.

$x^3+x^2+x-3=0$

$P(x)=x^3+x^2+x-3$

$P(1)=1^3+1^2+1-3=0$

$\therefore\ x=1$ ✓

$$\begin{array}{c|cccc} 1 & 1 & 1 & 1 & -3 \\ & \downarrow & 1 & 2 & 3 \\ \hline & 1 & 2 & 3 & 0 \end{array}$$

$\therefore P(x)=(x-1)(x^2+2x+3)$ ✓

$x=1$ or $x^2+2x+3=0$

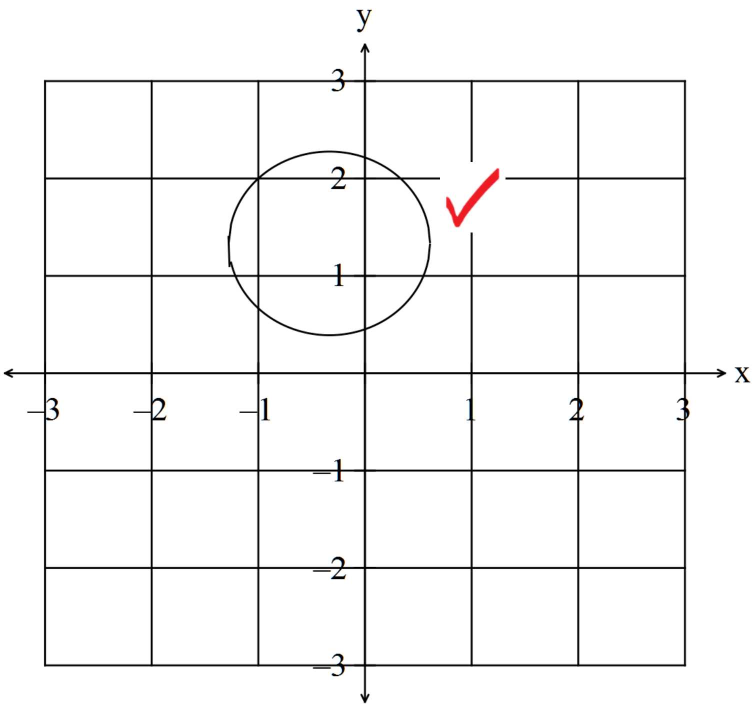
$$x=\frac{-2\pm\sqrt{4-12}}{2} \qquad -8=8i^2$$

$x=1$ or $x=\dfrac{-2\pm i2\sqrt{2}}{2}$

$x=1$ or $x=-1\pm i\sqrt{2}$ ✓

Question 3 **[5,3 = 8 marks]**

(a) Sketch $\{z : |x - 1 + iy| = 2|x + i(y-1)|\}$ on the set of axes below.



$$\sqrt{(x-1)^2 + y^2} = 2\sqrt{x^2 + (y-1)^2}$$

$$(x-1)^2 + y^2 = 4\left(x^2 + (y-1)^2\right)$$

$$x^2 - 2x + 1 + y^2 = 4\left(x^2 + y^2 - 2y + 1\right)$$

$$3x^2 + 3y^2 + 2x - 8y + 3 = 0$$

$$3\left(x^2 + y^2 + \frac{2x}{3} - \frac{8y}{3} + 1\right) = 0$$

$$C\left(-\frac{1}{3}, \frac{4}{3}\right) \quad r = \sqrt{\frac{1}{9} + \frac{16}{9} - 1} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

(b) If $z = \frac{1+i}{1-i} \times (3+3i)$ find the expression for $\bar{z}$.

$$z = 3\frac{1+i}{1-i} \times (1+i) \times \frac{1+i}{1+i}$$ ✓

$$= 3\frac{(1+3i+3i^2+i^3)}{1-i^2}$$ ✓

$$= \frac{3}{2}(1+3i-3-i)$$

$$z = \frac{3}{2}(-2+2i)$$

$$z = 3(-1+i)$$

$$\therefore \bar{z} = 3(-1-i)$$ ✓

Alternate Solution:

$$1+i = \sqrt{2}cis\frac{\pi}{4}, \quad 1-i = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$$

$$z = \frac{3(1+i)^2}{1-i}$$

$$z = \frac{3\left(\sqrt{2}cis\frac{\pi}{4}\right)^2}{\sqrt{2}cis\left(-\frac{\pi}{4}\right)}$$ ✓

$$z = \frac{3\times 2cis\frac{\pi}{2}}{\sqrt{2}cis\left(-\frac{\pi}{4}\right)}$$

$$z = 3\sqrt{2}cis\frac{3\pi}{4}$$ ✓

$$\bar{z} = 3\sqrt{2}cis\left(-\frac{3\pi}{4}\right)$$ ✓

Question 4 **[6,4 = 10 marks]**

(a) Use De Moivre's theorem to express $\cos(3x)$ in terms of $\cos(x)$.

$$(cos(x)+i\,sin(x))^3 = cos(3x)+i\,sin(3x)$$

so $cos(3x) = Re(cos(x)+i\,sin(x))^3$

$$(cos(x)+i\,sin(x))^3 = cos^3(x)+3i\,sin(x)cos^2(x)+3i^2\,sin^2(x)cos(x)+i^3\,sin^3(x)$$

$$(cos(x)+i\,sin(x))^3 = (cos^3(x)-3sin^2(x)cos(x))+i(3sin(x)cos^2(x)-sin^3(x))$$

as $cos(3x) = Re(cos(x)+i\,sin(x))^3$

we have $cos(3x) = Re(cos^3(x)-3sin^2(x)cos(x))+i(3sin(x)cos^2(x)-sin^3(x))$

so $cos(3x) = cos^3(x)-3sin^2(x)cos(x)$

$= cos^3(x)-3cos(x)[1-cos^2(x)]$

$= cos^3(x)-3cos(x)+3cos^3(x)$

Therefore $cos(3x) = 4cos^3(x)-3cos(x)$

(b) Calculate $(-1-i)^{10}$. Give your answer in cartesian form.

$$(-1-i)^{10} = \left(\sqrt{2}cis\left(-\frac{3\pi}{4}\right)\right)^{10}$$

$$= 2^5 cis\left(-\frac{30\pi}{4}\right)$$

$$= 32cis\left(-\frac{15\pi}{2}\right)$$

$$= 32cis\frac{\pi}{2}$$

$$= 32i$$

Question 5 **[4 marks]**

Determine the complex number $z = a + bi$, where a, b are real constants with $a > 0$ such that $Im\left(\frac{1}{z^2}\right) = \frac{1}{100}$ and $Im(z) = -2Re(z)$.

$Im(z) = -2Re(z) \quad \Rightarrow \quad b = -2a$

$\Rightarrow z = a - 2ai = a(1 - 2i)$ ✓

$z^2 = a^2(1 - 4i + 4i^2)$

$z^2 = a^2(-3 - 4i)$

$$\frac{1}{z^2} = \frac{1}{-a^2(3 + 4i)} \times \frac{3 - 4i}{3 - 41}$$

$$\frac{1}{z^2} = -\frac{1}{a^2}\left(\frac{3 - 4i}{25}\right)$$ ✓

$$\therefore Im\left(\frac{1}{z^2}\right) = \frac{4}{25a^2} = \frac{1}{100}$$

$a^2 = 16, \quad a > 0, \quad a = 4$ ✓

$\therefore b = -8$

$\therefore z = 4 - 8i$ ✓

Alternate Working (yet more effort)

$$\frac{1}{z^2} = \frac{a^2 - b^2}{(a^2 - b^2)^2 + (2ab)^2} - \frac{2abi}{(a^2 - b^2)^2 + (2ab)^2}$$ ✓

$$\therefore \frac{1}{100} = \frac{-2ab}{(a^2 - b^2)^2 + (2ab)^2}$$

Sub in $b = -2a \quad \Rightarrow \frac{1}{100} = \frac{-2a(-2a)}{(a^2 - (-2a)^2)^2 + (2a(-2a))^2}$ ✓

$\frac{1}{100} = \frac{4a^2}{9a^4 + 16a^4} \quad \Rightarrow a = 4$ ✓

$\Rightarrow b = -8 \quad \Rightarrow z = 4 - 8i$ ✓

Question 6 **[3 marks]**

Simplify the expression below.

$$\left(\frac{\sqrt{3}cis\frac{3\pi}{4}}{6cis\frac{5\pi}{6}cis\frac{2\pi}{3}}\right)^{-1}$$

$$=\frac{6cis\frac{5\pi}{6}cis\frac{2\pi}{3}}{\sqrt{3}cis\frac{3\pi}{4}}$$ ✓

$$=\frac{6cis\frac{3\pi}{2}}{\sqrt{3}cis\frac{3\pi}{4}}$$ ✓

$$=2\sqrt{3}cis\frac{3\pi}{4}$$ ✓

Question 7 **[2,3,2 = 7 marks]**

a) Given z is a complex number with modulus r and argument θ, express the modulus and argument of each of the complex numbers z_1 and z_2 in terms of r and θ where

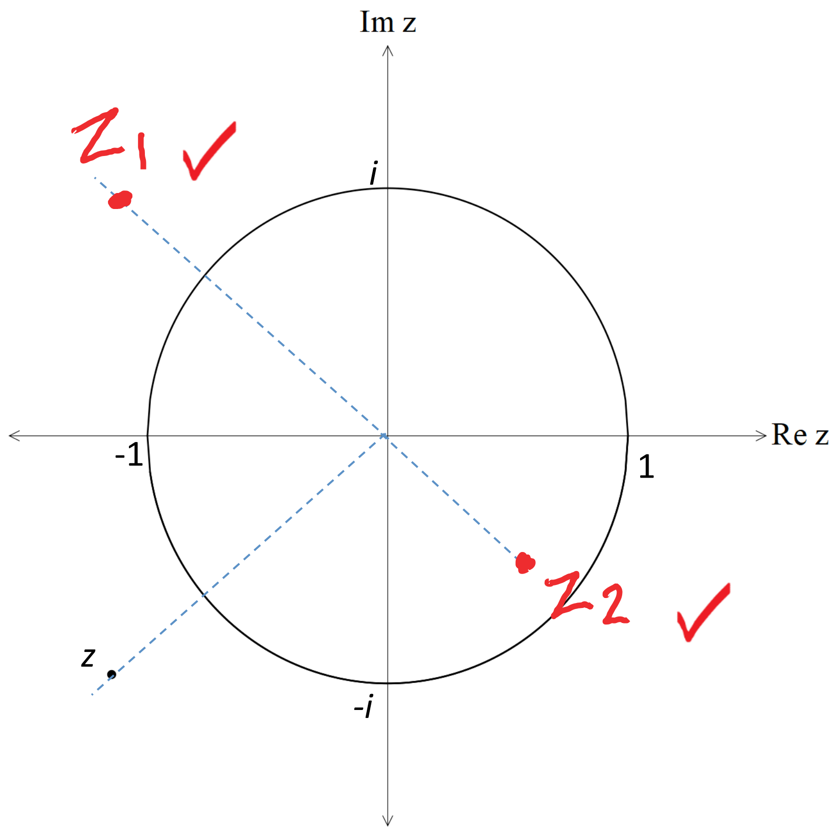
i) $z_1 = \bar{z}$.

$$|z_1| = r \qquad \arg(z_1) = -\theta$$

ii) $z_2 = -z^{-1}$.

$$|z_2| = \frac{1}{r} \qquad \arg(z^{-1}) = -\theta \qquad \arg(z_2) = \pi - \theta$$

b) The diagram below shows the circle in the complex plane and the position of the complex number z.



Given the approximate values of r and θ are 1.5 and $220°$ respectively, indicate the locations of he complex numbers z_1 and z_2 as defined in part (a) on the diagram above.

Question 8 **[3 marks]**

If w is any complex cube root of unity, simplify $(1+4w)(1+4w^2)$.

$$(1+4w)(1+4w^2) = 1 + 4w + 4w^2 + 16w^3$$

$$= 4 + 4w + 4w^2 + 16(1) - 3$$ ✓

Expanding and sub $w^3 = 1$

$$= 4(1 + w + w^2) + 13$$

$$= 4(0) + 13$$ ✓

$$= 13$$ ✓

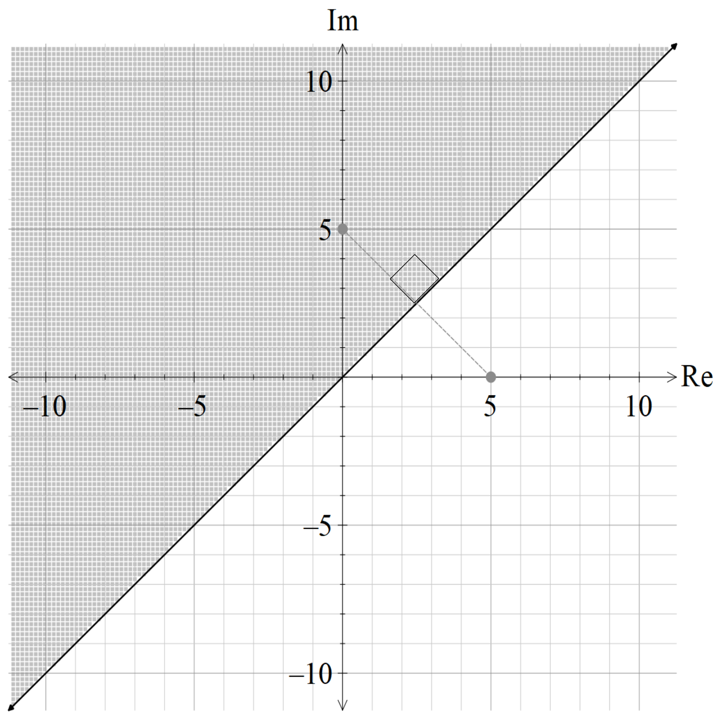
Isolating $1 + w + w^2$

Sub $1 + w + w^2 = 0$ and simplifying

Alternate method shown below Q 9.

Question 9 **[3 marks]**

Describe the locus of z in the following Argand Diagram.



Line: $|z - 5i| = |z - 5|$ ✓ ✓

Locus: $|z - 5i| \leq |z - 5|$ ✓

Alternate working:

$$w = cis\ \frac{2\pi}{3}, \quad \therefore w^2 = cis\ \frac{4\pi}{3} \text{ (or vice versa)}$$

$$\therefore w = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad w^2 == -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$ ✓

$$(1 + 4w)(1 + 4w^2) = \left(1 + 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\right)\left(1 + 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)\right)$$

$$= (1 - 2 + 2\sqrt{3}i)(1 - 2 - 2\sqrt{3}i)$$

$$= (-1 + 2\sqrt{3}i)(-1 - 2\sqrt{3}i)$$ ✓

$$= (-1)^2 + \left(2\sqrt{3}\right)^2$$

$$= 1 + 12$$

$$= 13$$ ✓

k